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Conformal symmetries of spacetimes

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Abstract

In this paper, we give a unified and global new approach to the study of the conformal structure of the three classical Riemannian spaces as well as of the six relativistic and non-relativistic spacetimes (Minkowskian, de Sitter, anti-de Sitter, and both Newton–Hooke and Galilean). We obtain general expressions within a Cayley–Klein framework, holding simultaneously for all these nine spaces, whose cycles (including geodesics and circles) are explicitly characterized in a new way. The corresponding cycle-preserving symmetries, which give rise to (Möbius-like) conformal Lie algebras, together with their differential realizations are then deduced without having to resort to solving the conformal Killing equations. We show that each set of three spaces with the same signature type and any curvature have isomorphic conformal algebras; these are related through an apparently new conformal duality. Laplace and wave-type differential equations with conformal algebra symmetry are finally constructed.

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1. Introduction

The role of conformal groups in physics can hardly be overestimated. Taking aside conformal invariance in quantum field theory, this role appears even at a rather basic level. In special relativity, spacetime is a flat pseudo-Riemannian space whose kinematical motion group is the Poincaré group $ISO(3,1)$; its 'angle'-preserving transformations generate the Minkowskian conformal group $SO(4,2)$ that contains as a subgroup the Poincaré group. This conformal group is the maximal invariance group of the vacuum Maxwell equations [1–4], and in general, of a large number of equations in different areas, for instance, all equations describing zero-mass systems [5–8].

The group of conformal transformations of the N-dimensional (ND) Euclidean space $\mathbf{E}^{\mathrm{N}}$ was first found by Liouville [9] and can be obtained from two equivalent approaches.

The 'conformal' method searches for transformations preserving the Euclidean metric up to proportionality; these preserve the angle between any two crossing curves. The 'hypersphere' method is to look for (local) transformations which carry hyperspheres into hyperspheres, including hyperplanes as limit cases [10]; it was introduced by Lie and Darboux. Both constructions lead to identical results except for the 2D case, where the conformal angle-preserving group is infinite dimensional, while the circle-preserving one has dimension 6. In the generic ND case, transformations obtained through either method generate a Lie group isomorphic to $SO_0(N+1,1)$ or to $O(N+1,1)$ when discrete reflections and inversions are also considered.

The Minkowskian and Euclidean spaces are two important particular instances within the family of *flat* homogeneous spaces $\mathbf{R}^{p,q} \equiv ISO(p,q)/SO(p,q)$. The conformal group of $\mathbf{R}^{p,q}$ is isomorphic to $SO_0(p+1,q+1)$ (discrete reflections and inversions not included) [9, 11]; its transformations preserve the 'angle' between any two curves and they carry $\mathbf{R}^{p,q}$-hyperspheres into themselves. In this context, it is rather natural to inquire which are the conformal groups of the Riemannian and pseudo-Riemannian spaces with *non-zero* constant curvature (for $N=2$, they are the sphere $\mathbf{S}^2$, the hyperbolic plane $\mathbf{H}^2$ and the two de Sitter 'spheres'), and also to analyse the conformal groups of the (contracted) cases with a degenerate metric (for $N=2$, these are the $(1+1)$D Galilean and the two Newton–Hooke spacetimes). The above spaces together with the Euclidean $\mathbf{E}^2$ and the Minkowskian $\mathbf{M}^{1+1}$ spaces constitute the nine Cayley–Klein (CK) spaces in two dimensions [12–15].

The aim of this paper is to present a new derivation of *cycle-preserving* (conformal) transformations which is much simpler than the usual ones and applies to any space(time), whether flat or curved and with any signature. The paper contains new results in relation to conformal groups of curved spacetimes, and collects an extensive amount of explicit information on the cycle-preserving transformations of the nine 2D CK spaces; furthermore, this approach may be generalized to higher dimensions.

In section 2, we describe the nine 2D CK spaces, whose metric structure is studied in section 3 by introducing three sets of geodesic coordinates. In section 4, we deduce the equations of *cycles* as lines with constant geodesic curvature that include geodesics, equidistants (or hypercycles), horocycles and circles. Next, in section 5, we obtain the groups of cycle-preserving symmetries, together with the differential realizations of their corresponding conformal algebras. Differential equations with conformal algebra symmetry are constructed in section 6. Some remarks end the paper.

2. The nine two-dimensional Cayley–Klein spaces

To begin with, we recall the algebraic structure of the nine 2D CK spaces [15]. Their motion groups are collectively denoted by $SO_{\kappa_1,\kappa_2}(3)$, where κ_1, κ_2 are two real coefficients, which can be reduced to +1, 0 and −1. The commutation relations of the CK algebra $so_{\kappa_1,\kappa_2}(3)$ on the basis $\{P_1, P_2, J_{12}\}$ and the Casimir invariant read

$$[J_{12},P_1]=P_2 \qquad [J_{12},P_2]=-\kappa_2 P_1 \qquad [P_1,P_2]=\kappa_1 J_{12} \tag{2.1}$$

$$\mathcal{C}=\kappa_2 P_1^2+P_2^2+\kappa_1 J_{12}^2. \tag{2.2}$$

The *space of points* corresponds to the 2D symmetric homogeneous space

$$S^2_{[\kappa_1],\kappa_2}=SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2) \qquad SO_{\kappa_2}(2)=\langle J_{12}\rangle \tag{2.3}$$

hence, the generator J_{12} leaves a point O (the origin) invariant, thus acting as the rotation around O, while P_1 and P_2 generate translations that move O along two basic directions.

Table 1. The nine two-dimensional CK spaces $S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$.

Elliptic: $\mathbf{S}^2$ $S^2_{[+],+} = SO(3)/SO(2)$	Euclidean: $\mathbf{E}^2$ $S^2_{[0],+} = ISO(2)/SO(2)$	Hyperbolic: $\mathbf{H}^2$ $S^2_{[-],+} = SO(2,1)/SO(2)$
Oscillating NH: $\mathbf{NH}^{1+1}_+$ (Co-Euclidean) $S^2_{[+],0} = ISO(2)/ISO(1)$	Galilean: $\mathbf{G}^{1+1}$ $S^2_{[0],0} = IISO(1)/ISO(1)$	Expanding NH: $\mathbf{NH}^{1+1}_-$ (Co-Minkowskian) $S^2_{[-],0} = ISO(1,1)/ISO(1)$
Anti-de Sitter: $\mathbf{AdS}^{1+1}$ (Co-hyperbolic) $S^2_{[+],-} = SO(2,1)/SO(1,1)$	Minkowskian: $\mathbf{M}^{1+1}$ $S^2_{[0],-} = ISO(1,1)/SO(1,1)$	De Sitter: $\mathbf{dS}^{1+1}$ (Doubly hyperbolic) $S^2_{[-],-} = SO(2,1)/SO(1,1)$

The space $S^2_{[\kappa_1],\kappa_2}$ has a canonical *metric* of *signature* diag$(1, \kappa_2)$ which turns out to have *constant curvature* κ_1. We display the nine 2D CK spaces in table 1; any vanishing coefficient κ_i can be interpreted as an Inönü–Wigner contraction and corresponds to either vanishing curvature ($\kappa_1 \to 0$) or degenerating metric ($\kappa_2 \to 0$).

Spacetimes with constant curvature [16] appear in this scheme. If $\{P_1, P_2, J_{12}\}$ are read as generators of time translations, space translations and boosts, respectively, the six CK groups with $\kappa_2 \leqslant 0$ (second and third rows of table 1) are the (kinematical) motion groups of $(1+1)$D spacetimes, where the coefficients κ_i are related to the universe time radius τ and speed of light c by

$$\kappa_1 = \pm 1/\tau^2 \qquad \kappa_2 = -1/c^2. \tag{2.4}$$

The curvature κ_1 may also be considered as a cosmological constant. Contractions $\kappa_1 \to 0$ and $\kappa_2 \to 0$ correspond to the flat limit $\tau \to \infty$ and to the non-relativistic limit $c \to \infty$, respectively. According to the values of (κ_1, κ_2) we find in table 1:

- three 'absolute-time' or non-relativistic spacetimes for $\kappa_2 = 0$: oscillating Newton–Hooke $\mathbf{NH}^{1+1}_+$ $(+, 0)$, Galilean $\mathbf{G}^{1+1}$ $(0, 0)$ and expanding Newton–Hooke $\mathbf{NH}^{1+1}_-$ $(-, 0)$ (we denote $ISO(1) \equiv \mathbb{R}$), with a degenerate Riemannian metric of signature diag$(+, 0)$;
- three 'relative-time' spacetimes for $\kappa_2 < 0$: anti-de Sitter $\mathbf{AdS}^{1+1}$ $(+, -)$, Minkowskian $\mathbf{M}^{1+1}$ $(0, -)$ and de Sitter $\mathbf{dS}^{1+1}$ $(-, -)$, with a Lorentzian metric of signature diag$(+, -)$.

A 3D real matrix representation of $so_{\kappa_1,\kappa_2}(3)$ is given by

$$P_1 = -\kappa_1 e_{01} + e_{10} \qquad P_2 = -\kappa_1\kappa_2 e_{02} + e_{20} \qquad J_{12} = -\kappa_2 e_{12} + e_{21} \tag{2.5}$$

where e_{ij} is a 3D matrix with a single non-zero entry 1 at row i and column j ($i, j = 0, 1, 2$). The exponential of these matrices leads to one-parametric subgroups of $SO_{\kappa_1,\kappa_2}(3)$,

$$\begin{aligned} &\mathrm{e}^{\alpha P_1} = \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{e}^{\gamma J_{12}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix} \\ &\mathrm{e}^{\beta P_2} = \begin{pmatrix} C_{\kappa_1\kappa_2}(\beta) & 0 & -\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1\kappa_2}(\beta) & 0 & C_{\kappa_1\kappa_2}(\beta) \end{pmatrix} \end{aligned} \tag{2.6}$$

where we have introduced the cosine $C_\kappa(x)$ and sine $S_\kappa(x)$ functions [13–15]

$$C_\kappa(x) = \begin{cases} \cos \sqrt{\kappa}\, x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh \sqrt{-\kappa}\, x & \kappa < 0 \end{cases} \qquad S_\kappa(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa}\, x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa}\, x & \kappa < 0. \end{cases} \tag{2.7}$$

From them, we define the 'versed sine' (or versine) $V_\kappa(x)$ and the tangent $T_\kappa(x)$,

$$V_\kappa(x) = \frac{1}{\kappa}(1 - C_\kappa(x)) \qquad T_\kappa(x) = \frac{S_\kappa(x)}{C_\kappa(x)}. \tag{2.8}$$

These curvature-dependent functions coincide with the circular and hyperbolic ones for $\kappa = \pm 1$; the contracted case $\kappa = 0$ gives rise to the parabolic or Galilean functions: $C_0(x) = 1$, $S_0(x) = x$ and $V_0(x) = x^2/2$. Identities for these functions can be found in the appendix of [15]. We display their derivatives where the corresponding inverse functions [14] are denoted by the prefix 'arc-',

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}C_\kappa(x) = -\kappa S_\kappa(x) && \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arc} C_\kappa(x) = \frac{-1}{\kappa\sqrt{\frac{1}{\kappa}(1-x^2)}}\\
&\frac{\mathrm{d}}{\mathrm{d}x}S_\kappa(x) = C_\kappa(x) && \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arc} S_\kappa(x) = \frac{1}{\sqrt{1-\kappa x^2}}\\
&\frac{\mathrm{d}}{\mathrm{d}x}T_\kappa(x) = \frac{1}{C_\kappa^2(x)} && \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arc} T_\kappa(x) = \frac{1}{1+\kappa x^2}\\
&\frac{\mathrm{d}}{\mathrm{d}x}V_\kappa(x) = S_\kappa(x) && \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arc} V_\kappa(x) = \frac{1}{\sqrt{2x-\kappa x^2}}.
\end{aligned} \tag{2.9}$$

By taking into account the realization (2.6), the CK group $SO_{\kappa_1,\kappa_2}(3)$ can be seen as a group of linear transformations in an ambient space $\mathbb{R}^3 = (x^0, x^1, x^2)$, acting as the group of isometries of a bilinear form $\Lambda = \operatorname{diag}(1, \kappa_1, \kappa_1\kappa_2)$. Therefore, an element $X \in SO_{\kappa_1,\kappa_2}(3)$ satisfies $X^{\mathrm{T}}\Lambda X = \Lambda$ where X^{T} denotes the transpose matrix of X.

The action of $SO_{\kappa_1,\kappa_2}(3)$ on $\mathbb{R}^3$ is linear but not transitive, since it conserves the quadratic form $(x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2$ provided by Λ, and $SO_{\kappa_2}(2) = \langle J_{12}\rangle$ is the isotropy subgroup of the point $O \equiv (1, 0, 0)$ which will be taken as the *origin* in the space $S^2_{[\kappa_1],\kappa_2}$. The action becomes transitive if we restrict ourselves to the orbit in $\mathbb{R}^3$ of the point O, which is contained in the 'sphere' Σ,

$$\Sigma \equiv (x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 = 1. \tag{2.10}$$

This orbit is identified with the CK space $S^2_{[\kappa_1],\kappa_2}$, and (x^0, x^1, x^2) fulfilling (2.10) are called *Weierstrass coordinates*; these allow us to obtain a differential realization of the generators as first-order vector fields in $\mathbb{R}^3$ with $\partial_i = \partial/\partial x^i$,

$$P_1 = \kappa_1 x^1\partial_0 - x^0\partial_1 \qquad P_2 = \kappa_1\kappa_2 x^2\partial_0 - x^0\partial_2 \qquad J_{12} = \kappa_2 x^2\partial_1 - x^1\partial_2. \tag{2.11}$$

3. Metric structure and coordinate systems

If both coefficients κ_i are different from zero, $SO_{\kappa_1,\kappa_2}(3)$ is a simple Lie group, and the space $S^2_{[\kappa_1],\kappa_2}$ is endowed with a non-degenerate metric g_0 coming from the non-singular Killing–Cartan form in the Lie algebra $so_{\kappa_1,\kappa_2}(3)$. At the origin, g_0 is given by

$$g_0(P_1, P_1) = -2\kappa_1 \qquad g_0(P_2, P_2) = -2\kappa_1\kappa_2 \qquad g_0(P_1, P_2) = 0. \tag{3.1}$$

To cover the cases with $\kappa_1 = 0$ where g_0 vanishes identically, we take a factor $-2\kappa_1$ out of g_0, and introduce the space *main metric* g_1 as $-2g_1 := g_0/\kappa_1$. If $\kappa_2 = 0$, g_1 is a degenerate metric and the action of $SO_{\kappa_1,0}(3)$ on $S^2_{[\kappa_1],0}$ has an invariant foliation. The restriction of g_1 to each foliation leaf vanishes, but $g_2 = \frac{1}{\kappa_2}g_1$ has a non-vanishing and well-defined restriction to each leaf; we call g_2 the *subsidiary metric*. A unified description of the metric structure for the nine 2D CK spaces is summed up as follows [14].

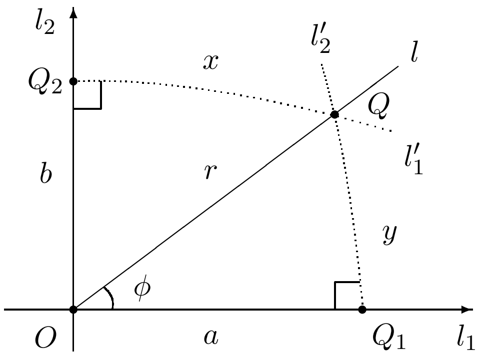

Figure 1. The three geodesic coordinate systems (a, y), (x, b) and (r, ϕ) of a point Q. The geodesics l'_1 and l'_2 are orthogonal to l_2 and l_1 through Q with intersection points Q_2 and Q_1, respectively.

Proposition 1. *The metric structure for a generic space $S^2_{[\kappa_1],\kappa_2}$ is characterized by:*

- *A connection ∇ which is invariant under $SO_{\kappa_1,\kappa_2}(3)$.*
- *A hierarchy of two metrics g_1 and $g_2 = \frac{1}{\kappa_2} g_1$ compatible with ∇. The action of $SO_{\kappa_1,\kappa_2}(3)$ on $S^2_{[\kappa_1],\kappa_2}$ is by isometries of both metrics.*
- *The main metric g_1 is actually a metric in the true sense and has constant curvature κ_1 and signature* $\mathrm{diag}(+, \kappa_2)$.
- *If $\kappa_2 \neq 0$, g_2 is a true quadratic metric proportional to g_1. If $\kappa_2 = 0$, the subsidiary metric g_2 gives a true metric only in each leaf of the invariant foliation in $S^2_{[\kappa_1],0}$, whose set of leaves can be parametrized by $(x^0)^2 + \kappa_1(x^1)^2 = 1 \equiv S^1_{[\kappa_1]}$; g_2 has signature* $(+)$.

In terms of Weierstrass coordinates in the linear ambient space $\mathbb{R}^3$, the two metrics in $S^2_{[\kappa_1],\kappa_2}$ come from the flat ambient metric

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 + \kappa_1(\mathrm{d}x^1)^2 + \kappa_1\kappa_2(\mathrm{d}x^2)^2 \tag{3.2}$$

in the form

$$(\mathrm{d}s^2)_1 = \frac{1}{\kappa_1}\,\mathrm{d}s^2 \qquad (\mathrm{d}s^2)_2 = \frac{1}{\kappa_2}(\mathrm{d}s^2)_1. \tag{3.3}$$

Now we proceed to introduce three relevant coordinate systems of geodesic type in $S^2_{[\kappa_1],\kappa_2}$ (see figure 1). Let us consider the origin $O \equiv (1, 0, 0)$, two (oriented) geodesics l_1 and l_2 which are orthogonal through the origin and a generic point Q with Weierstrass coordinates $\mathbf{x} = (x^0, x^1, x^2)$. By taking into account (2.6), we have

- If $\mathbf{x} = \exp(aP_1)\exp(yP_2)O$, we call (a, y) the *type I geodesic parallel coordinates of* Q.
- If $\mathbf{x} = \exp(bP_2)\exp(xP_1)O$, we call (x, b) the *type II geodesic parallel coordinates of* Q.
- The *geodesic polar coordinates* of the point Q are (r, ϕ) if $\mathbf{x} = \exp(\phi J_{12})\exp(rP_1)O$.

Weierstrass coordinates $\mathbf{x}$ of a generic point Q are displayed in table 2 in three geodesic coordinate systems; by substitution in the expressions of the metrics in Weierstrass coordinates, (3.2) and (3.3), we find the main and subsidiary metrics in any of the geodesic coordinates. From them, we may compute the connection symbols Γ^i_{jk}. The area element $\mathrm{d}S$ in coordinates, say u^1 and u^2, is $\sqrt{\det g_1/\kappa_2}\,\mathrm{d}u^1 \wedge \mathrm{d}u^2$; this information is also summarized in table 2. When the pair (κ_1, κ_2) is particularized to its nine essentially different values, relations appearing in table 2 provide the explicit description of the nine spaces $S^2_{[\kappa_1],\kappa_2}$. This is illustrated

Table 2. Weierstrass coordinates, metric, canonical connection and area element for $S^2_{[\kappa_1],\kappa_2}$ given in the three geodesic coordinate systems.

Parallel I (a, y)	Parallel II (x, b)	Polar (r, ϕ)
$x^0 = C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)$	$x^0 = C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)$	$x^0 = C_{\kappa_1}(r)$
$x^1 = S_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)$	$x^1 = S_{\kappa_1}(x)$	$x^1 = S_{\kappa_1}(r)C_{\kappa_2}(\phi)$
$x^2 = S_{\kappa_1\kappa_2}(y)$	$x^2 = C_{\kappa_1}(x)S_{\kappa_1\kappa_2}(b)$	$x^2 = S_{\kappa_1}(r)S_{\kappa_2}(\phi)$
$(\mathrm{d}s^2)_1 = C^2_{\kappa_1\kappa_2}(y)\,\mathrm{d}a^2 + \kappa_2\,\mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}x^2 + \kappa_2 C^2_{\kappa_1}(x)\,\mathrm{d}b^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}r^2 + \kappa_2 S^2_{\kappa_1}(r)\,\mathrm{d}\phi^2$
$(\mathrm{d}s^2)_2 = \mathrm{d}y^2$, for $a = a_0$	$(\mathrm{d}s^2)_2 = C^2_{\kappa_1}(x)\,\mathrm{d}b^2$, for $x = x_0$	$(\mathrm{d}s^2)_2 = S^2_{\kappa_1}(r)\,\mathrm{d}\phi^2$, for $r = r_0$
$\Gamma^y_{aa} = \kappa_1 S_{\kappa_1\kappa_2}(y)C_{\kappa_1\kappa_2}(y)$	$\Gamma^x_{bb} = \kappa_1\kappa_2 S_{\kappa_1}(x)C_{\kappa_1}(x)$	$\Gamma^r_{\phi\phi} = -\kappa_2 S_{\kappa_1}(r)C_{\kappa_1}(r)$
$\Gamma^a_{ay} = -\kappa_1\kappa_2 T_{\kappa_1\kappa_2}(y)$	$\Gamma^b_{bx} = -\kappa_1 T_{\kappa_1}(x)$	$\Gamma^\phi_{\phi r} = 1/T_{\kappa_1}(r)$
$\mathrm{d}\mathcal{S} = C_{\kappa_1\kappa_2}(y)\,\mathrm{d}a \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = C_{\kappa_1}(x)\,\mathrm{d}x \wedge \mathrm{d}b$	$\mathrm{d}\mathcal{S} = S_{\kappa_1}(r)\,\mathrm{d}r \wedge \mathrm{d}\phi$

Table 3. The nine spaces $S^2_{[\kappa_1],\kappa_2}$ in geodesic parallel I coordinates (a, y). In the three Riemannian spaces $\kappa_1 \in \{1, 0, -1\}$ and $\kappa_2 = 1$. In the six spacetimes, $\kappa_1 = \pm 1/\tau^2$, $\kappa_2 = -1/c^2$, $a \equiv t$ is the time coordinate and y is the space one.

$\mathbf{S}^2 = S^2_{[+],+}$	$\mathbf{E}^2 = S^2_{[0],+}$	$\mathbf{H}^2 = S^2_{[-],+}$
$x^0 = \cos a \cos y$	$x^0 = 1$	$x^0 = \cosh a \cosh y$
$x^1 = \sin a \cos y$	$x^1 = a$	$x^1 = \sinh a \cosh y$
$x^2 = \sin y$	$x^2 = y$	$x^2 = \sinh y$
$(\mathrm{d}s^2)_1 = \cos^2 y\,\mathrm{d}a^2 + \mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}a^2 + \mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \cosh^2 y\,\mathrm{d}a^2 + \mathrm{d}y^2$
$\Gamma^y_{aa} = \sin y \cos y$	$\Gamma^y_{aa} = 0$	$\Gamma^y_{aa} = -\sinh y \cosh y$
$\Gamma^a_{ay} = -\tan y$	$\Gamma^a_{ay} = 0$	$\Gamma^a_{ay} = \tanh y$
$\mathrm{d}\mathcal{S} = \cos y\,\mathrm{d}a \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}a \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \cosh y\,\mathrm{d}a \wedge \mathrm{d}y$
$\mathbf{NH}^{1+1}_+ = S^2_{[+1/\tau^2],0}$	$\mathbf{G}^{1+1} = S^2_{[0],0}$	$\mathbf{NH}^{1+1}_- = S^2_{[-1/\tau^2],0}$
$x^0 = \cos(t/\tau)$	$x^0 = 1$	$x^0 = \cosh(t/\tau)$
$x^1 = \tau \sin(t/\tau)$	$x^1 = t$	$x^1 = \tau \sinh(t/\tau)$
$x^2 = y$	$x^2 = y$	$x^2 = y$
$(\mathrm{d}s^2)_1 = \mathrm{d}t^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}t^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}t^2$
$(\mathrm{d}s^2)_2 = \mathrm{d}y^2$, $t = t_0$	$(\mathrm{d}s^2)_2 = \mathrm{d}y^2$, $t = t_0$	$(\mathrm{d}s^2)_2 = \mathrm{d}y^2$, $t = t_0$
$\Gamma^y_{tt} = \frac{1}{\tau^2}y,\ \Gamma^t_{ty} = 0$	$\Gamma^y_{tt} = 0,\ \Gamma^t_{ty} = 0$	$\Gamma^y_{tt} = -\frac{1}{\tau^2}y,\ \Gamma^t_{ty} = 0$
$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$
$\mathbf{AdS}^{1+1} = S^2_{[+1/\tau^2],-1/c^2}$	$\mathbf{M}^{1+1} = S^2_{[0],-1/c^2}$	$\mathbf{dS}^{1+1} = S^2_{[-1/\tau^2],-1/c^2}$
$x^0 = \cos(t/\tau)\cosh(y/c\tau)$	$x^0 = 1$	$x^0 = \cosh(t/\tau)\cos(y/c\tau)$
$x^1 = \tau \sin(t/\tau)\cosh(y/c\tau)$	$x^1 = t$	$x^1 = \tau \sinh(t/\tau)\cos(y/c\tau)$
$x^2 = c\tau \sinh(y/c\tau)$	$x^2 = y$	$x^2 = c\tau \sin(y/c\tau)$
$(\mathrm{d}s^2)_1 = \cosh^2(y/c\tau)\,\mathrm{d}t^2 - \frac{1}{c^2}\,\mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}t^2 - \frac{1}{c^2}\,\mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \cos^2(y/c\tau)\,\mathrm{d}t^2 - \frac{1}{c^2}\,\mathrm{d}y^2$
$\Gamma^y_{tt} = \frac{c}{\tau}\sinh(y/c\tau)\cosh(y/c\tau)$	$\Gamma^y_{tt} = 0$	$\Gamma^y_{tt} = -\frac{c}{\tau}\sin(y/c\tau)\cos(y/c\tau)$
$\Gamma^t_{ty} = \frac{1}{c\tau}\tanh(y/c\tau)$	$\Gamma^t_{ty} = 0$	$\Gamma^t_{ty} = -\frac{1}{c\tau}\tan(y/c\tau)$
$\mathrm{d}\mathcal{S} = \cosh(y/c\tau)\,\mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \cos(y/c\tau)\,\mathrm{d}t \wedge \mathrm{d}y$

in parallel I coordinates in table 3. Note that the non-relativistic (Newtonian) spacetimes with $c = \infty$ ($\kappa_2 = 0$) have foliations with leaves defined by constant $a = t$, and the subsidiary (space) metric is relevant only within them. We find a degenerate main temporal metric $(\mathrm{d}s^2)_1 = \mathrm{d}t^2$ (providing 'absolute-time'), and an invariant foliation whose leaves are the 'absolute-space' at the moment $t = t_0$ with a non-degenerate subsidiary spatial metric $(\mathrm{d}s^2)_2 = \mathrm{d}y^2$ defined in each leaf.

4. Cycles

Cycles are defined as lines with constant geodesic curvature. Within the vector model in the ambient space $\mathbb{R}^3$, it can be shown that:

Proposition 2. *In a 2D CK space $S^2_{[\kappa_1],\kappa_2}$ with non-zero curvature $\kappa_1 \neq 0$, cycles are the intersections of the 'sphere' Σ (2.10) with planes in $\mathbb{R}^3$, that is,*

$$\alpha_0 x^0 + \alpha_1 x^1 + \alpha_2 x^2 = \alpha \tag{4.1}$$

where $\alpha, \alpha_0, \alpha_1$ and α_2 are constants; the geodesic curvature k_g of the corresponding cycle is

$$k_g^2 = \kappa_1^2 \alpha^2 \frac{\kappa_2}{\alpha_2^2 + \kappa_2 \alpha_1^2 + \kappa_1 \kappa_2 \left(\alpha_0^2 - \alpha^2\right)}. \tag{4.2}$$

Now we deduce the cycle equations in the geodesic coordinate systems. We will assume $\kappa_1 \neq 0$, but the final results will also hold when $\kappa_1 = 0$. Let us consider geodesic parallel I coordinates. The cycle equation, obtained by introducing (a, y) of table 2 in (4.1), reads

$$\alpha_0 C_{\kappa_1}(a) + \alpha_1 S_{\kappa_1}(a) = \frac{\alpha}{C_{\kappa_1\kappa_2}(y)} - \alpha_2 T_{\kappa_1\kappa_2}(y) \tag{4.3}$$

which can be recast in another much simpler and unknown form by means of relations (A.8) involving the lambda funcion $\Lambda_{\kappa_1\kappa_2}(y) \equiv y^\wedge$ of y described in the appendix,

$$\alpha_0 C_{\kappa_1}(a) + \alpha_1 S_{\kappa_1}(a) = \alpha C_{-\kappa_1\kappa_2}(y^\wedge) - \alpha_2 S_{-\kappa_1\kappa_2}(y^\wedge). \tag{4.4}$$

In the same way, we find the equations of cycles in the remaining geodesic coordinate systems; these are summarized in table 4. In each system, we obtain several equivalent expressions for the rhs of the cycle equations. In particular, the last form in polar coordinates can be rewritten as a quadratic equation in $T_{\kappa_1}(r/2)$,

$$T^2_{\kappa_1}(r/2) - \frac{1}{\kappa_1}\frac{2}{\alpha + \alpha_0}(\alpha_1 C_{\kappa_2}(\phi) + \alpha_2 S_{\kappa_2}(\phi)) T_{\kappa_1}(r/2) + \frac{1}{\kappa_1}\frac{\alpha - \alpha_0}{\alpha + \alpha_0} = 0. \tag{4.5}$$

Let $T_{\kappa_1}(r_1/2)$, $T_{\kappa_1}(r_2/2)$ be the two roots of this equation, where r_1 and r_2 are the distances from the origin O to the two intersection points of the cycle with the line $\phi =$ constant through O. Then the product

$$T_{\kappa_1}(r_1/2) T_{\kappa_1}(r_2/2) = \frac{1}{\kappa_1}\frac{\alpha - \alpha_0}{\alpha + \alpha_0} =: \wp \tag{4.6}$$

turns out to be the same for all lines through O. In the flat limit this reduces to $r_1 r_2 = 4\wp$, so that the quantity $\wp$ (or rather $4\wp$) should be called the *power of the point relative to the cycle*.

In what follows, we proceed to identify the equations of the three *generic* types of cycles: *geodesics* (zero geodesic curvature), *equidistants* (constant distance to a geodesic) and *circles* (constant distance to a point), which are also displayed in table 4.

4.1. Geodesics

If we set $\alpha = 0$ in (4.3), then $k_g = 0$ and we obtain *geodesics* as intersections of Σ with planes through the origin in $\mathbb{R}^3$ in the vector model. For $\alpha_2 = 0$ we have the non-generic geodesics $a = a_0$; the generic ones arise when $\alpha_2 \neq 0$ by setting $\beta_0 = -\alpha_0/\alpha_2$ and $\beta_1 = -\alpha_1/\alpha_2$,

$$T_{\kappa_1\kappa_2}(y) = \beta_0 C_{\kappa_1}(a) + \beta_1 S_{\kappa_1}(a). \tag{4.7}$$

In the relativistic spacetimes ($\kappa_2 = -1/c^2 < 0$), geodesic (4.7) can be either a time-like, space-like or isotropic one according to the character of its tangent vector, distinguished by the sign either >0, <0, $= 0$ of $1 + \kappa_1\kappa_2\beta_0^2 + \kappa_2\beta_1^2$, respectively. Hereafter, when dealing with geodesics we will distinguish the possible types; this is an *actual* distinction only whenever $\kappa_2 < 0$ but is irrelevant in the Riemannian spaces with $\kappa_2 > 0$, where all geodesics merge in a single type.

Table 4. Equations of cycles in the spaces $S^2_{[\kappa_1],\kappa_2}$ in the geodesic coordinate systems. The circle equations give the finite (time-like) distance between two points, and the equidistant equations determine the (time-like) distance between a point and a (space-like) line.

	Geodesic parallel I coordinates (a, y)
Cycles	$\alpha_0 C_{\kappa_1}(a) + \alpha_1 S_{\kappa_1}(a) = \dfrac{\alpha}{C_{\kappa_1\kappa_2}(y)} - \alpha_2 T_{\kappa_1\kappa_2}(y) = \alpha C_{-\kappa_1\kappa_2}(y^\wedge) - \alpha_2 S_{-\kappa_1\kappa_2}(y^\wedge)$
	$= \Big(\dfrac{\alpha\sqrt{\kappa_1\kappa_2} - \alpha_2}{2\sqrt{\kappa_1\kappa_2}}\Big)\Big(\dfrac{1 + \sqrt{\kappa_1\kappa_2}T_{\kappa_1\kappa_2}(y/2)}{1 - \sqrt{\kappa_1\kappa_2}T_{\kappa_1\kappa_2}(y/2)}\Big) + \Big(\dfrac{\alpha\sqrt{\kappa_1\kappa_2} + \alpha_2}{2\sqrt{\kappa_1\kappa_2}}\Big)\Big(\dfrac{1 - \sqrt{\kappa_1\kappa_2}T_{\kappa_1\kappa_2}(y/2)}{1 + \sqrt{\kappa_1\kappa_2}T_{\kappa_1\kappa_2}(y/2)}\Big)$
Geodesics	$T_{\kappa_1\kappa_2}(y) = \beta_0 C_{\kappa_1}(a) + \beta_1 S_{\kappa_1}(a)$ and $a = a_0$
Equidistants	$S^2_{\kappa_1}(d) = \kappa_2 \dfrac{\Big\{S_{\kappa_1\kappa_2}(y) - C_{\kappa_1\kappa_2}(y)\Big(\beta_0 C_{\kappa_1}(a) + \beta_1 S_{\kappa_1}(a)\Big)\Big\}^2}{1 + \kappa_2\beta_1^2 + \kappa_1\kappa_2\beta_0^2}$ $\quad (\kappa_2 \neq 0)$
Circles	$C_{\kappa_1}(\rho) = C_{\kappa_1\kappa_2}(y)C_{\kappa_1\kappa_2}(y_0)C_{\kappa_1}(a - a_0) + \kappa_1\kappa_2 S_{\kappa_1\kappa_2}(y)S_{\kappa_1\kappa_2}(y_0)$
	$V_{\kappa_1}(\rho) = C_{\kappa_1\kappa_2}(y)C_{\kappa_1\kappa_2}(y_0)V_{\kappa_1}(a - a_0) + \kappa_2 V_{\kappa_1\kappa_2}(y - y_0)$
	Geodesic parallel II coordinates (x, b)
Cycles	$\alpha_0 C_{\kappa_1\kappa_2}(b) + \alpha_2 S_{\kappa_1\kappa_2}(b) = \dfrac{\alpha}{C_{\kappa_1}(x)} - \alpha_1 T_{\kappa_1}(x) = \alpha C_{-\kappa_1}(x^\wedge) - \alpha_1 S_{-\kappa_1}(x^\wedge)$
	$= \Big(\dfrac{\alpha\sqrt{\kappa_1} - \alpha_1}{2\sqrt{\kappa_1}}\Big)\Big(\dfrac{1 + \sqrt{\kappa_1}T_{\kappa_1}(x/2)}{1 - \sqrt{\kappa_1}T_{\kappa_1}(x/2)}\Big) + \Big(\dfrac{\alpha\sqrt{\kappa_1} + \alpha_1}{2\sqrt{\kappa_1}}\Big)\Big(\dfrac{1 - \sqrt{\kappa_1}T_{\kappa_1}(x/2)}{1 + \sqrt{\kappa_1}T_{\kappa_1}(x/2)}\Big)$
Geodesics	$T_{\kappa_1}(x) = \beta_0 C_{\kappa_1\kappa_2}(b) + \beta_2 S_{\kappa_1\kappa_2}(b)$ and $b = b_0$
Equidistants	$S^2_{\kappa_1}(d) = \kappa_2 \dfrac{\Big\{S_{\kappa_1}(x) - C_{\kappa_1}(x)\Big(\beta_0 C_{\kappa_1\kappa_2}(b) + \beta_2 S_{\kappa_1\kappa_2}(b)\Big)\Big\}^2}{\kappa_2 + \beta_2^2 + \kappa_1\kappa_2\beta_0^2}$ $\quad (\kappa_2 \neq 0)$
Circles	$C_{\kappa_1}(\rho) = C_{\kappa_1}(x)C_{\kappa_1}(x_0)C_{\kappa_1\kappa_2}(b - b_0) + \kappa_1 S_{\kappa_1}(x)S_{\kappa_1}(x_0)$
	$V_{\kappa_1}(\rho) = \kappa_2 C_{\kappa_1}(x)C_{\kappa_1}(x_0)V_{\kappa_1\kappa_2}(b - b_0) + V_{\kappa_1}(x - x_0)$
	Geodesic polar coordinates (r, ϕ)
Cycles	$\alpha_1 C_{\kappa_2}(\phi) + \alpha_2 S_{\kappa_2}(\phi) = \dfrac{\alpha}{S_{\kappa_1}(r)} - \dfrac{\alpha_0}{T_{\kappa_1}(r)} = \kappa_1\Big(\dfrac{\alpha + \alpha_0}{2}\Big)T_{\kappa_1}(r/2) + \Big(\dfrac{\alpha - \alpha_0}{2}\Big)\dfrac{1}{T_{\kappa_1}(r/2)}$
Geodesics	$\dfrac{1}{T_{\kappa_1}(r)} = \beta_1 C_{\kappa_2}(\phi) + \beta_2 S_{\kappa_2}(\phi)$ and $\phi = \phi_0$
Equidistants	$S^2_{\kappa_1}(d) = \kappa_2 \dfrac{\Big\{C_{\kappa_1}(r) - S_{\kappa_1}(r)(\beta_1 C_{\kappa_2}(\phi) + \beta_2 S_{\kappa_2}(\phi))\Big\}^2}{\beta_2^2 + \kappa_2\beta_1^2 + \kappa_1\kappa_2}$ $\quad (\kappa_2 \neq 0)$
Circles	$C_{\kappa_1}(\rho) = C_{\kappa_1}(r)C_{\kappa_1}(r_0) + \kappa_1 S_{\kappa_1}(r)S_{\kappa_1}(r_0)C_{\kappa_2}(\phi - \phi_0)$
	$V_{\kappa_1}(\rho) = V_{\kappa_1}(r - r_0) + \kappa_2 S_{\kappa_1}(r)S_{\kappa_1}(r_0)V_{\kappa_2}(\phi - \phi_0)$

4.2. Equidistants

In the relativistic spacetimes, the equidistant to a given geodesic is of the same type as the geodesic, which can also be either time-like or space-like, but its equidistance radius is space-like or time-like, respectively. The two branches of the *equidistant* to a 'space-like' line given by (4.7) with 'time-like' equidistance radius d are

$$S^2_{\kappa_1}(d) = \kappa_2 \frac{\big\{S_{\kappa_1\kappa_2}(y) - C_{\kappa_1\kappa_2}(y)(\beta_0 C_{\kappa_1}(a) + \beta_1 S_{\kappa_1}(a))\big\}^2}{1 + \kappa_2\beta_1^2 + \kappa_1\kappa_2\beta_0^2} \qquad \text{for} \quad \kappa_2 \neq 0. \tag{4.8}$$

These are the particular cycles with equation (4.3) for $\alpha_0 = -\beta_0\alpha_2$, $\alpha_1 = -\beta_1\alpha_2$ and $\alpha_2 \neq 0$, as given for the baseline geodesic, but with

$$\alpha = \pm\alpha_2 S_{\kappa_1}(d)\left(\beta_1^2 + \kappa_1\beta_0^2 + 1/\kappa_2\right)^{1/2} \tag{4.9}$$

instead of $\alpha = 0$, and their geodesic curvature (4.2) reads

$$k_g = \big|\kappa_1 T_{\kappa_1}(d)\big|. \tag{4.10}$$

Table 5. Particularized equations for geodesics (besides $a = a_0$) and circles (constant geodesic 'time-like' distance ρ to a fixed centre (a_0, y_0)) for the nine spaces $S^2_{[\kappa_1],\kappa_2}$ in geodesic parallel I coordinates and with the same conventions as in table 3.

$\mathbf{S}^2 = S^2_{[+],+}$	$\mathbf{E}^2 = S^2_{[0],+}$	$\mathbf{H}^2 = S^2_{[-],+}$
$\tan y = \beta_0 \cos a + \beta_1 \sin a$	$y = \beta_0 + \beta_1 a$	$\tanh y = \beta_0 \cosh a + \beta_1 \sinh a$
$\cos\rho = \cos y \cos y_0 \cos(a - a_0)$ $+ \sin y \sin y_0$	$\rho^2 = (a - a_0)^2$ $+ (y - y_0)^2$	$\cosh\rho = \cosh y \cosh y_0 \cosh(a - a_0)$ $- \sinh y \sinh y_0$
$\mathbf{NH}^{1+1}_+ = S^2_{[+1/\tau^2],0}$	$\mathbf{G}^{1+1} = S^2_{[0],0}$	$\mathbf{NH}^{1+1}_- = S^2_{[-1/\tau^2],0}$
$y = \beta_0 \cos(t/\tau) + \beta_1 \tau \sin(t/\tau)$	$y = \beta_0 + \beta_1 t$	$y = \beta_0 \cosh(t/\tau) + \beta_1 \tau \sinh(t/\tau)$
$\cos(\rho/\tau) = \cos((t - t_0)/\tau)$	$\rho^2 = (t - t_0)^2$	$\cosh(\rho/\tau) = \cosh((t - t_0)/\tau)$
$\mathbf{AdS}^{1+1} = S^2_{[+1/\tau^2],-1/c^2}$	$\mathbf{M}^{1+1} = S^2_{[0],-1/c^2}$	$\mathbf{dS}^{1+1} = S^2_{[-1/\tau^2],-1/c^2}$
$c\tau \tanh(y/c\tau) = \beta_0 \cos(t/\tau) + \beta_1 \tau \sin(t/\tau)$	$y = \beta_0 + \beta_1 t$	$c\tau \tan(y/c\tau) = \beta_0 \cosh(t/\tau) + \beta_1 \tau \sinh(t/\tau)$
$\cos(\rho/\tau) = -\sinh(y/c\tau) \sinh(y_0/c\tau)$ $+ \cosh(y/c\tau) \cosh(y_0/c\tau) \cos((t - t_0)/\tau)$	$\rho^2 = (t - t_0)^2$ $- \frac{1}{c^2}(y - y_0)^2$	$\cosh(\rho/\tau) = \sin(y/c\tau) \sin(y_0/c\tau)$ $+ \cos(y/c\tau) \cos(y_0/c\tau) \cosh((t - t_0)/\tau)$

4.3. Circles

A 'space-like' *circle* of centre (a_0, y_0) and 'time-like' radius ρ is

$$C_{\kappa_1}(\rho) = C_{\kappa_1\kappa_2}(y) C_{\kappa_1\kappa_2}(y_0) C_{\kappa_1}(a - a_0) + \kappa_1\kappa_2 S_{\kappa_1\kappa_2}(y) S_{\kappa_1\kappa_2}(y_0) \tag{4.11}$$

and can, alternatively, be written in terms of versed sines (2.8),

$$V_{\kappa_1}(\rho) = C_{\kappa_1\kappa_2}(y) C_{\kappa_1\kappa_2}(y_0) V_{\kappa_1}(a - a_0) + \kappa_2 V_{\kappa_1\kappa_2}(y - y_0). \tag{4.12}$$

These equations are equivalent to the one in the form (4.3) for the following choice of α_i:

$$\begin{aligned} &\alpha_0 = C_{\kappa_1}(a_0) C_{\kappa_1\kappa_2}(y_0) \qquad && \alpha_1 = \kappa_1 S_{\kappa_1}(a_0) C_{\kappa_1\kappa_2}(y_0) \\ &\alpha_2 = \kappa_1\kappa_2 S_{\kappa_1\kappa_2}(y_0) \qquad && \alpha = C_{\kappa_1}(\rho). \end{aligned} \tag{4.13}$$

Hence, the geodesic curvature (4.2) for a circle reads

$$k_g = |1/T_{\kappa_1}(\rho)| \tag{4.14}$$

reducing to $k_g = 1/\rho$ in the flat case. The circle equations also give the finite form of the *distance* $s \equiv \rho$ between two points.

The different types of 'space-like' cycles so obtained can be classified according to the sign of κ_1 and the value of the geodesic curvature for each specific CK space is as follows.

- If $\kappa_1 > 0$ ($\mathbf{S}^2$ and $\mathbf{AdS}^{1+1}$), geodesic curvatures of 'space-like' equidistants (4.10) and 'space-like' circles (4.14) range in $k_g \in (0, \infty)$, so equidistants (4.8) are at the same time circles (4.11) and vice versa.
- If $\kappa_1 = 0$ ($\mathbf{E}^2$ and $\mathbf{M}^{1+1}$), equidistants are simply geodesics with $k_g = 0$, while circles correspond to $k_g \in (0, \infty)$.
- If $\kappa_1 < 0$ ($\mathbf{H}^2$ and $\mathbf{dS}^{1+1}$), 'space-like' equidistants (4.8) and circles (4.11) are *different* cycles and a third type, naturally separating them, appears; these are the 'space-like' *horocycles*, which are the common limits of equidistants when the 'space-like' base geodesic goes to infinity or of 'space-like' circles when the centre goes to infinity. In this case, equidistants (or hypercycles) correspond to the values of $k_g \in (0, \sqrt{-\kappa_1})$ while circles have $k_g \in (\sqrt{-\kappa_1}, \infty)$, with $k_g = 0$ for geodesics, $k_g = \sqrt{-\kappa_1}$ for horocycles and $k_g = \infty$ for circles of radius 0.

In table 5, we explicitly write the equations of geodesics and circles for each of the nine spaces $S^2_{[\kappa_1],\kappa_2}$ in parallel I coordinates (a study of Galilean cycles can be found in [12]).

5. Cycle-preserving Lie groups

In this section, we give a completely new derivation of cycle-preserving groups in $S^2_{[\kappa_1],\kappa_2}$, which leads to some illuminating views on the relation between conformal algebras for curved and flat spaces.

5.1. One-parameter subgroups of geodesic-preserving transformations

We start by (re)deriving the geodesic-preserving transformations. Consider the equations of geodesics in parallel I coordinates of table 4. We look whether there exists a transformation within the ansatz $(a, y) \to (a'(a), y)$ which carries geodesics into geodesics. Noting that y is assumed not to change, this requirement is equivalent to imposing

$$\beta_0' C_{\kappa_1}(a'(a)) + \beta_1' S_{\kappa_1}(a'(a)) \propto \beta_0 C_{\kappa_1}(a) + \beta_1 S_{\kappa_1}(a). \tag{5.1}$$

The addition properties of the trigonometric functions display, directly, the obvious solution

$$(a, y) \to (a + a_0, y) \tag{5.2}$$

which is the *translation* along the geodesic l_1 of $S^2_{[\kappa_1],\kappa_2}$ generated by P_1. Now we start with the ansatz $(x, b) \to (x, b'(b))$ in parallel II coordinates; similar arguments lead to *another* geodesic-preserving map,

$$(x, b) \to (x, b + b_0) \tag{5.3}$$

which is the *translation* along the geodesic l_2 generated by P_2. Finally, we look within the ansatz $(r, \phi) \to (r, \phi'(\phi))$ in polar coordinates; the condition of preserving geodesics singles out the transformation

$$(r, \phi) \to (r, \phi + \phi_0) \tag{5.4}$$

which is the *rotation* around the origin of $S^2_{[\kappa_1],\kappa_2}$ with the generator J_{12}. The fundamental vector fields of three one-parameter groups (5.2)–(5.4) are given by

$$P_1 = -\partial_a \qquad P_2 = -\partial_b \qquad J_{12} = -\partial_\phi. \tag{5.5}$$

By writing them in a single coordinate system (see table 6) it can be shown that they close a Lie algebra with commutators (2.1); thus, we recover the CK algebra $so_{\kappa_1,\kappa_2}(3)$.

5.2. One-parameter subgroups of cycle-preserving transformations

We now look for additional cycle-preserving transformations which are not geodesic-preserving. A natural idea is to consider three ansatzes complementary to those previously explored, that is, $(a, y) \to (a, y'(y))$, $(x, b) \to (x'(x), b)$ and $(r, \phi) \to (r'(r), \phi)$.

Let us begin with the ansatz $(a, y) \to (a, y'(y))$ and require this to be cycle-preserving. By taking (4.4) into account and noting that a is assumed not to change, we enforce

$$\alpha' C_{-\kappa_1\kappa_2}(y'^{\wedge}) - \alpha_2' S_{-\kappa_1\kappa_2}(y'^{\wedge}) \propto \alpha C_{-\kappa_1\kappa_2}(y^{\wedge}) - \alpha_2 S_{-\kappa_1\kappa_2}(y^{\wedge}) \tag{5.6}$$

and again, the addition trigonometric relations display a solution $y'^{\wedge} = y^{\wedge} + \xi$, which is a one-parametric subgroup of transformations with canonical parameter ξ and whose fundamental vector field follows using (A.8):

$$L_2 = -\partial_{y^{\wedge}} = -C_{\kappa_1\kappa_2}(y)\partial_y. \tag{5.7}$$

Transformations generated by L_2 behave as translations in the neighbourhood of the origin O; we call L_2 the generator of Λ-*translations* along the geodesic l_2 of $S^2_{[\kappa_1],\kappa_2}$. If one now takes into account (A.7), then Λ-translations can be rewritten as

$$(a, y) \to (a, y'(y)) \qquad y' = \Lambda_{-\kappa_1\kappa_2}\left(\Lambda_{\kappa_1\kappa_2}(y) + \xi\right). \tag{5.8}$$

Table 6. Parametrized differential realizations for the generators $\{P_i, J_{12}, D, L_i, G_i, R_i\}$ of conformal algebras $\mathrm{conf}_{\kappa_1,\kappa_2}$ (these are zero-realizations for both conformal Casimirs).

Geodesic parallel I coordinates (a, y)

$$P_1 = -\partial_a,\ P_2 = -\kappa_1\kappa_2 S_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\partial_a - C_{\kappa_1}(a)\partial_y$$

$$J_{12} = \kappa_2 C_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\partial_a - S_{\kappa_1}(a)\partial_y,\ D = -\frac{S_{\kappa_1}(a)}{C_{\kappa_1\kappa_2}(y)}\partial_a - C_{\kappa_1}(a) S_{\kappa_1\kappa_2}(y)\partial_y$$

$$L_1 = -\frac{C_{\kappa_1}(a)}{C_{\kappa_1\kappa_2}(y)}\partial_a + \kappa_1 S_{\kappa_1}(a) S_{\kappa_1\kappa_2}(y)\partial_y,\ L_2 = -C_{\kappa_1\kappa_2}(y)\partial_y$$

$$G_1 = \frac{1}{C_{\kappa_1\kappa_2}(y)}(V_{\kappa_1}(a) - \kappa_2 V_{\kappa_1\kappa_2}(y))\partial_a + S_{\kappa_1}(a) S_{\kappa_1\kappa_2}(y)\partial_y$$

$$G_2 = \kappa_2 S_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\partial_a - (V_{\kappa_1}(a) - \kappa_2 V_{\kappa_1\kappa_2}(y))\partial_y$$

$$R_1 = -\frac{1}{2}\left(1 + \frac{C_{\kappa_1}(a)}{C_{\kappa_1\kappa_2}(y)}\right)\partial_a + \frac{1}{2}\kappa_1 S_{\kappa_1}(a) S_{\kappa_1\kappa_2}(y)\partial_y$$

$$R_2 = -\tfrac{1}{2}\kappa_1\kappa_2 S_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\partial_a - \tfrac{1}{2}(C_{\kappa_1}(a) + C_{\kappa_1\kappa_2}(y))\partial_y$$

Geodesic parallel II coordinates (x, b)

$$P_1 = -C_{\kappa_1\kappa_2}(b)\partial_x - \kappa_1 T_{\kappa_1}(x) S_{\kappa_1\kappa_2}(b)\partial_b,\ P_2 = -\partial_b$$

$$J_{12} = \kappa_2 S_{\kappa_1\kappa_2}(b)\partial_x - T_{\kappa_1}(x) C_{\kappa_1\kappa_2}(b)\partial_b,\ D = -S_{\kappa_1}(x) C_{\kappa_1\kappa_2}(b)\partial_x - \tfrac{S_{\kappa_1\kappa_2}(b)}{C_{\kappa_1}(x)}\partial_b$$

$$L_1 = -C_{\kappa_1}(x)\partial_x,\ L_2 = \kappa_1\kappa_2 S_{\kappa_1}(x) S_{\kappa_1\kappa_2}(b)\partial_x - \frac{C_{\kappa_1\kappa_2}(b)}{C_{\kappa_1}(x)}\partial_b$$

$$G_1 = \Big(V_{\kappa_1}(x) - \kappa_2 V_{\kappa_1\kappa_2}(b)\Big)\partial_x + T_{\kappa_1}(x) S_{\kappa_1\kappa_2}(b)\partial_b$$

$$G_2 = \kappa_2 S_{\kappa_1}(x) S_{\kappa_1\kappa_2}(b)\partial_x - \frac{1}{C_{\kappa_1}(x)}\Big(V_{\kappa_1}(x) - \kappa_2 V_{\kappa_1\kappa_2}(b)\Big)\partial_b$$

$$R_1 = -\tfrac{1}{2}\Big(C_{\kappa_1}(x) + C_{\kappa_1\kappa_2}(b)\Big)\partial_x - \tfrac{1}{2}\kappa_1 T_{\kappa_1}(x) S_{\kappa_1\kappa_2}(b)\partial_b$$

$$R_2 = \frac{1}{2}\kappa_1\kappa_2 S_{\kappa_1}(x) S_{\kappa_1\kappa_2}(b)\partial_x - \frac{1}{2}\left(1 + \frac{C_{\kappa_1\kappa_2}(b)}{C_{\kappa_1}(x)}\right)\partial_b$$

Geodesic polar coordinates (r, ϕ)

$$P_1 = -C_{\kappa_2}(\phi)\partial_r + \frac{S_{\kappa_2}(\phi)}{T_{\kappa_1}(r)}\partial_\phi,\ P_2 = -\kappa_2 S_{\kappa_2}(\phi)\partial_r - \frac{C_{\kappa_2}(\phi)}{T_{\kappa_1}(r)}\partial_\phi$$

$$J_{12} = -\partial_\phi,\ D = -S_{\kappa_1}(r)\partial_r$$

$$L_1 = -C_{\kappa_2}(\phi) C_{\kappa_1}(r)\partial_r + \frac{S_{\kappa_2}(\phi)}{S_{\kappa_1}(r)}\partial_\phi,\ L_2 = -\kappa_2 S_{\kappa_2}(\phi) C_{\kappa_1}(r)\partial_r - \frac{C_{\kappa_2}(\phi)}{S_{\kappa_1}(r)}\partial_\phi$$

$$G_1 = C_{\kappa_2}(\phi) V_{\kappa_1}(r)\partial_r + S_{\kappa_2}(\phi)\frac{V_{\kappa_1}(r)}{S_{\kappa_1}(r)}\partial_\phi$$

$$G_2 = \kappa_2 S_{\kappa_2}(\phi) V_{\kappa_1}(r)\partial_r - C_{\kappa_2}(\phi)\frac{V_{\kappa_1}(r)}{S_{\kappa_1}(r)}\partial_\phi$$

$$R_1 = -\frac{1}{2T_{\kappa_1}(r/2)}\Big(C_{\kappa_2}(\phi) S_{\kappa_1}(r)\partial_r - S_{\kappa_2}(\phi)\partial_\phi\Big)$$

$$R_2 = -\frac{1}{2T_{\kappa_1}(r/2)}\Big(\kappa_2 S_{\kappa_2}(\phi) S_{\kappa_1}(r)\partial_r + C_{\kappa_2}(\phi)\partial_\phi\Big)$$

Likewise, the transformation

$$(x, b) \to (x'(x), b) \qquad x' = \Lambda_{-\kappa_1}\left(\Lambda_{\kappa_1}(x) + \zeta\right) \tag{5.9}$$

with the canonical parameter ζ is also cycle-preserving, and its fundamental vector field is the generator of Λ-*translations* along the geodesic l_1 of $S^2_{[\kappa_1],\kappa_2}$,

$$L_1 = -\partial_{x^\wedge} = -C_{\kappa_1}(x)\partial_x. \tag{5.10}$$

The search for a cycle-preserving transformation within $(r, \phi) \to (r'(r), \phi)$ remains. Using the cycle equation in the second form given in table 4, introducing the power $\wp$ of the origin relative to cycle (4.6) and taking into account that ϕ is assumed not to change, it suffices to enforce

$$\frac{1}{\wp'}T_{\kappa_1}(r'/2) + \frac{1}{T_{\kappa_1}(r'/2)} \propto \frac{1}{\wp}T_{\kappa_1}(r/2) + \frac{1}{T_{\kappa_1}(r/2)}. \qquad (5.11)$$

Hence, the one-parametric family of transformations with canonical parameter λ,

$$T_{\kappa_1}(r'/2) = \mathrm{e}^{\lambda}T_{\kappa_1}(r/2) \qquad (5.12)$$

is cycle-preserving as it verifies (5.11) with $\wp' = \mathrm{e}^{2\lambda}\wp$; it can be interpreted as 'homoteties' around the origin with scale factor e^{λ}. The fundamental vector field is obtained with the aid of (2.9)

$$D = -S_{\kappa_1}(r)\partial_r. \qquad (5.13)$$

We note the natural appearance of a one-parameter family of *dilations* in spaces such as the sphere, hyperbolic plane and both de Sitter spacetimes, with non-zero curvature, in spite of a vague but widespread belief against this possibility.

We must stress that L_1, L_2 generate *cycle-preserving* transformations for curved spaces. When $\kappa_1 = 0$, the approach discussed in this paragraph leads only to the standard dilation $r' = \mathrm{e}^{\lambda}r$ as a new transformation beyond the motions; generators L_i are still meaningful in the flat case $\kappa_1 = 0$ but they *coincide* with P_i and do not provide any new transformation. Nevertheless, the flat conformal algebras will be obtained below as some suitable limit.

5.3. Involutive discrete cycle-preserving transformations

Equations (5.6) or (5.11) allow other discrete cycle-preserving transformations not belonging to one-parametric families. These solutions correspond to *inversions in cycles* and are involutive. There are three families, each matching perfectly with the three earlier types D, L_1 and L_2. We start with the *inversions in circles*, associated with D. The discrete transformation

$$T_{\kappa_1}(r'/2)\cdot T_{\kappa_1}(r/2) = \wp_0 \qquad (5.14)$$

is also a solution of (5.11) with $\wp' = \wp = \wp_0$. The circle with centre at the origin and radius ρ_0 such that $T^2_{\kappa_1}(\rho_0/2) = -\wp_0$ is invariant under this transformation, which therefore corresponds to the inversion in that circle. For any value of κ_1, the product of two inversions in two concentric circles in the family (5.14) with constants $\wp_0$, $\wp'_0$ is a *dilation* (5.12) with the same centre and scale factor $\mathrm{e}^{\lambda} = \wp'_0/\wp_0$. All this is well known for the $\kappa_1 = 0$ case, where (5.14) reduces to the standard flat inversion in circles $r' = 4\wp_0/r$.

In the generic $\kappa_1 \neq 0$ case, both Λ-translations, (5.8) and (5.9), can also be expressed as a product of two involutive discrete transformations, namely *inversions in equidistants*. Specifically, the rhs of the cycle equations in parallel II coordinates can be recast in the third form given in table 4. If the condition of carrying cycles into cycles is enforced, this gives an equation fully analogous to (5.11), where the rhs is replaced by

$$\frac{1}{\kappa_1\wp_2}\left(\frac{1-\sqrt{\kappa_1}T_{\kappa_1}(x/2)}{1+\sqrt{\kappa_1}T_{\kappa_1}(x/2)}\right) + \left(\frac{1-\sqrt{\kappa_1}T_{\kappa_1}(x/2)}{1+\sqrt{\kappa_1}T_{\kappa_1}(x/2)}\right)^{-1} \qquad \wp_2 = \frac{1}{\kappa_1}\frac{\alpha\sqrt{\kappa_1}-\alpha_1}{\alpha\sqrt{\kappa_1}+\alpha_1} \qquad (5.15)$$

and the lhs is formally similar with α', α'_1, x' and $\wp'_2$. In this form, the equation allows us to read directly a discrete and involutive cycle-preserving transformation similar to (5.14),

$$\left(\frac{1-\sqrt{\kappa_1}T_{\kappa_1}(x'/2)}{1+\sqrt{\kappa_1}T_{\kappa_1}(x'/2)}\right)\cdot\left(\frac{1-\sqrt{\kappa_1}T_{\kappa_1}(x/2)}{1+\sqrt{\kappa_1}T_{\kappa_1}(x/2)}\right) = \kappa_1\wp_2. \qquad (5.16)$$

This transformation keeps invariant one branch of a certain equidistant (with parameters $\alpha_0 = \beta_0 = 0, \alpha_1 \neq 0, \alpha_2 = \beta_2 = 0$ and $\alpha = \alpha_1 S_{\kappa_1}(d)$) to the line l_2, and the 'time-like' equidistance d coincides with the x coordinate which is constant along the equidistant. The quantity $\wp_2$ may be called the *power of the baseline relative to that cycle*. Again, the product of two such inversions in two coaxial equidistants (with the same baseline) is a Λ-translation along the line l_1. Everything is similar for the third family corresponding to the cycle equations in parallel I coordinates, as described by analogous expressions with the replacements $\alpha \to \alpha$, $\alpha_1 \to \alpha_2$, $y \to x$, $\kappa_1 \to \kappa_1\kappa_2$ and $\wp_2 \to \wp_1$.

Thus, these *discrete inversions* are even more basic than the one-parameter transformations generated by D, L_1 and L_2, and they display a behaviour which in the generic curved case is completely symmetric between the three basic one-parameter motion subgroups; this symmetry disappears in the flat limit $\kappa_1 = 0$, where inversions reduce to ordinary reflections in geodesics (recall L_i reduce in this limit to P_i).

5.4. Conformal algebras

Summing up, for $\kappa_1 \neq 0$ we have found six one-parametric subgroups of cycle-preserving transformations in $S^2_{[\kappa_1],\kappa_2}$ with generators $\{P_i, J_{12}, L_i, D\}$ $(i = 1, 2)$. By writing them in the same coordinate system as shown in table 6, it can be checked that they close a Lie algebra denoted $\mathrm{conf}_{\kappa_1,\kappa_2}$ with Lie brackets and Casimir invariants given by

$$
\begin{array}{lll}
[J_{12}, P_1] = P_2 & [J_{12}, P_2] = -\kappa_2 P_1 & [P_1, P_2] = \kappa_1 J_{12} \\
[J_{12}, L_1] = L_2 & [J_{12}, L_2] = -\kappa_2 L_1 & [L_1, L_2] = -\kappa_1 J_{12} \\
[D, P_i] = L_i & [D, L_i] = P_i & [D, J_{12}] = 0 \\
[P_1, L_1] = \kappa_1 D & [P_2, L_2] = \kappa_1\kappa_2 D & \\
[P_1, L_2] = 0 & [P_2, L_1] = 0 &
\end{array}
\tag{5.17}
$$

$$
\begin{aligned}
\mathcal{C}_1' &= -\kappa_1 J_{12}^2 + \kappa_1\kappa_2 D^2 + \kappa_2\left(L_1^2 - P_1^2\right) + \left(L_2^2 - P_2^2\right) \\
\mathcal{C}_2' &= \kappa_1 J_{12} D + (L_1 P_2 - P_1 L_2).
\end{aligned}
\tag{5.18}
$$

An interesting trait, which, as far as we know has not been pointed out previously, is a *conformal duality* between the generators of translations and Λ-translations,

$$
P_i \leftrightarrow L_i \qquad J_{12} \leftrightarrow J_{12} \qquad D \leftrightarrow D
\tag{5.19}
$$

which interchanges the set of conformal algebras $\mathrm{conf}_{\kappa_1,\kappa_2} \leftrightarrow \mathrm{conf}_{-\kappa_1,\kappa_2}$ and thus, relates the conformal algebras of spaces with *opposite* curvatures and the same signature type.

Now we discuss the flat limit $\kappa_1 \to 0$. In this case, L_i coincide with the translation generators P_i, but as long as $\kappa_1 \neq 0$, we may take two other generators

$$
G_i = \frac{1}{\kappa_1}(L_i - P_i) \qquad i = 1, 2
\tag{5.20}
$$

which are always defined and continue to be independent of the four remaining generators P_i, J_{12} and D, when $\kappa_1 = 0$. In the Euclidean case, it turns out that G_i generate the so-called *specific conformal transformations*, so that we will keep this name in the general curved case. The differential realization of generators $\{P_i, J_{12}, G_i, D\}$ for any value of κ_1 is given in table 6. On this basis, the commutation rules and Casimirs of $\mathrm{conf}_{\kappa_1,\kappa_2}$ read

$$
\begin{array}{lll}
[J_{12}, P_1] = P_2 & [J_{12}, P_2] = -\kappa_2 P_1 & [P_1, P_2] = \kappa_1 J_{12} \\
[J_{12}, G_1] = G_2 & [J_{12}, G_2] = -\kappa_2 G_1 & [G_1, G_2] = 0 \\
[D, P_i] = P_i + \kappa_1 G_i & [D, G_i] = -G_i & [D, J_{12}] = 0 \\
[P_1, G_1] = D & [P_2, G_2] = \kappa_2 D & \\
[P_1, G_2] = -J_{12} & [P_2, G_1] = J_{12} &
\end{array}
\tag{5.21}
$$

$$\mathcal{C}_1 = -J_{12}^2 + \kappa_2 D^2 + \kappa_2(P_1G_1 + G_1P_1) + (P_2G_2 + G_2P_2) + \kappa_1\left(\kappa_2 G_1^2 + G_2^2\right)$$
$$\mathcal{C}_2 = J_{12}D + (G_1P_2 - P_1G_2) \qquad (5.22)$$

provided that $\mathcal{C}_i = \mathcal{C}_i'/\kappa_1$. Note that $\{P_1, P_2, J_{12}\}$ spans a CK algebra $so_{\kappa_1,\kappa_2}(3)$, but the set $\{P_1, P_2, J_{12}; D\}$ only closes a Lie algebra (isometries plus dilations) if $\kappa_1 = 0$.

Another interesting basis for $\mathrm{conf}_{\kappa_1,\kappa_2}$ is $\{R_i, J_{12}, G_i, D\}$, where

$$R_i = P_i + \tfrac{1}{2}\kappa_1 G_i = \tfrac{1}{2}(P_i + L_i) \qquad i = 1, 2. \qquad (5.23)$$

The commutation relations and Casimir operators are now given by

$$\begin{array}{lll} [J_{12}, R_1] = R_2 & [J_{12}, R_2] = -\kappa_2 R_1 & [R_1, R_2] = 0 \\ [J_{12}, G_1] = G_2 & [J_{12}, G_2] = -\kappa_2 G_1 & [G_1, G_2] = 0 \\ [D, R_i] = R_i & [D, G_i] = -G_i & [D, J_{12}] = 0 \\ [R_1, G_1] = D & [R_2, G_2] = \kappa_2 D & \\ [R_1, G_2] = -J_{12} & [R_2, G_1] = J_{12} & \end{array} \qquad (5.24)$$

$$\mathcal{C}_1 = -J_{12}^2 + \kappa_2 D^2 + \kappa_2(R_1G_1 + G_1R_1) + R_2G_2 + G_2R_2$$
$$\mathcal{C}_2 = J_{12}D + G_1R_2 - R_1G_2. \qquad (5.25)$$

Therefore, the curvature κ_1 disappears from the commutators; this is due to the fact that spaces with the same metric signature but opposite curvatures have essentially the same conformal algebra and was already suggested by the conformal duality $P_i \leftrightarrow L_i$ as this changed sign to κ_1. This means that *all* spaces in the family $S^2_{[\kappa_1],\kappa_2}$ with the *same* κ_2 have *isomorphic* conformal algebras. These are

- $so(3, 1)$ ((2+1))D de Sitter algebra) as the conformal algebra of the three 2D Riemannian spaces with $\kappa_2 > 0$,
- $iso(2, 1)$ ((2+1)D Poincaré) for the (1+1)D non-relativistic spacetimes with $\kappa_2 = 0$ and
- $so(2, 2)$ ((2+1)D anti-de Sitter) for the (1+1)D relativistic spacetimes with $\kappa_2 < 0$.

In relation to the usual approach to conformal groups, by solving the conformal Killing equations [9] we state the following:

Proposition 3. *All the vector fields X displayed in table 6 satisfy the conformal Killing equations for the metrics g_1, g_2 of the space $S^2_{[\kappa_1],\kappa_2}$, that is, $L_X g_i = \mu_X g_i$, where $L_X g_i$ is the Lie derivative of g_i. In Weierstrass coordinates, the conformal factors μ_X are given by*

$$\mu_{P_1} = \mu_{P_2} = \mu_{J_{12}} = 0 \qquad \mu_D = -2x^0 \qquad \mu_{G_1} = 2x^1 \qquad \mu_{G_2} = 2\kappa_2 x^2. \qquad (5.26)$$

6. Conformal symmetries of Laplace/wave-type equations

As a byproduct of the conformal vector fields deduced in the previous section, we now proceed to obtain differential equations with conformal algebra symmetry.

Let us consider a 2D space with coordinates (u^1, u^2), a differential operator $E = E(u^1, u^2, \partial_1, \partial_2)$ acting on functions $\Phi(u^1, u^2)$ defined on the space ($\partial_i \equiv \partial/\partial u^i$), and consider the differential equation

$$E\Phi(u^1, u^2) = 0. \qquad (6.1)$$

An operator $\mathcal{O}$ is a symmetry of (6.1) if $\mathcal{O}$ transforms solutions into solutions,

$$E\mathcal{O} = \mathcal{Q}E \qquad \text{or} \qquad [E, \mathcal{O}] = \mathcal{Q}'E \qquad (6.2)$$

Table 7. The Laplace–Beltrami operator $\mathcal{C}$ giving rise to differential Laplace and wave-type equations $\mathcal{C}\Phi = 0$ in geodesic parallel I $\Phi(a,y) \equiv \Phi(t,y)$ and polar $\Phi(r,\phi) \equiv \Phi(r,\chi)$ coordinates for the nine CK spaces (when $\kappa_2 \leqslant 0$ the angle is denoted as χ and is a rapidity in the kinematical interpretation).

$so(3,1): \mathbf{S}^2 = S^2_{[+],+}$	$so(3,1): \mathbf{E}^2 = S^2_{[0],+}$	$so(3,1): \mathbf{H}^2 = S^2_{[-],+}$
$\frac{1}{\cos^2 y}\partial_a^2 + \partial_y^2 - \tan y\,\partial_y$	$\partial_a^2 + \partial_y^2$	$\frac{1}{\cosh^2 y}\partial_a^2 + \partial_y^2 + \tanh y\,\partial_y$
$\frac{1}{\sin^2 r}\partial_\phi^2 + \partial_r^2 + \frac{1}{\tan r}\partial_r$	$\frac{1}{r^2}\partial_\phi^2 + \partial_r^2 + \frac{1}{r}\partial_r$	$\frac{1}{\sinh^2 r}\partial_\phi^2 + \partial_r^2 + \frac{1}{\tanh r}\partial_r$
$iso(2,1): \mathbf{NH}_+^{1+1} = S^2_{[+1/\tau^2],0}$	$iso(2,1): \mathbf{G}^{1+1} = S^2_{[0],0}$	$iso(2,1): \mathbf{NH}_-^{1+1} = S^2_{[-1/\tau^2],0}$
∂_y^2	∂_y^2	∂_y^2
$\frac{1}{\tau^2\sin^2(r/\tau)}\partial_\chi^2$	$\frac{1}{r^2}\partial_\chi^2$	$\frac{1}{\tau^2\sinh^2(r/\tau)}\partial_\chi^2$
$so(2,2): \mathbf{AdS}^{1+1} = S^2_{[+1/\tau^2],-1/c^2}$	$so(2,2): \mathbf{M}^{1+1} = S^2_{[0],-1/c^2}$	$so(2,2): \mathbf{dS}^{1+1} = S^2_{[-1/\tau^2],-1/c^2}$
$\frac{-1}{c^2\cosh^2(y/c\tau)}\partial_t^2 + \partial_y^2 + \frac{\tanh(y/c\tau)}{c\tau}\partial_y$	$-\frac{1}{c^2}\partial_t^2 + \partial_y^2$	$\frac{-1}{c^2\cos^2(y/c\tau)}\partial_t^2 + \partial_y^2 - \frac{\tan(y/c\tau)}{c\tau}\partial_y$
$\frac{1}{\tau^2\sin^2(r/\tau)}\partial_\chi^2 - \frac{1}{c^2}\partial_r^2 - \frac{1}{c^2\tau\tan(r/\tau)\partial_r}$	$\frac{1}{r^2}\partial_\chi^2 - \frac{1}{c^2}\partial_r^2 - \frac{1}{c^2 r}\partial_r$	$\frac{1}{\tau^2\sinh^2(r/\tau)}\partial_\chi^2 - \frac{1}{c^2}\partial_r^2 - \frac{1}{c^2\tau\tanh(r/\tau)}\partial_r$

where $\mathcal{Q}$ is another operator and $\mathcal{Q}' = \mathcal{Q} - \mathcal{O}$. We now focus our attention on the differential equation obtained by taking as E the Casimir $\mathcal{C}$ of the CK algebra $so_{\kappa_1,\kappa_2}(3)$ (2.2) in the space $S^2_{[\kappa_1],\kappa_2}$: $\mathcal{C}\Phi = 0$. In the three geodesic coordinate systems, such an equation turns out to be

$$\begin{aligned}
&\left(\frac{\kappa_2}{C^2_{\kappa_1\kappa_2}(y)}\partial_a^2 + \partial_y^2 - \kappa_1\kappa_2 T_{\kappa_1\kappa_2}(y)\partial_y\right)\Phi(a,y) = 0\\
&\left(\kappa_2\partial_x^2 - \kappa_1\kappa_2 T_{\kappa_1}(x)\partial_x + \frac{1}{C^2_{\kappa_1}(x)}\partial_b^2\right)\Phi(x,b) = 0\\
&\left(\kappa_2\partial_r^2 + \frac{\kappa_2}{T_{\kappa_1}(r)}\partial_r + \frac{1}{S^2_{\kappa_1}(r)}\partial_\phi^2\right)\Phi(r,\phi) = 0.
\end{aligned} \tag{6.3}$$

The conformal algebra $\mathrm{conf}_{\kappa_1,\kappa_2}$ is a symmetry algebra of these equations, and by using table 6, the generators $\{P_i, J_{12}, G_i, D\}$ are shown to fulfil relations (6.2) with the *same* factors appearing in (5.26),

$$\begin{aligned}
&[\mathcal{C}, X] = 0 \qquad X \in \{P_1, P_2, J_{12}\}\\
&[\mathcal{C}, D] = -2C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)\mathcal{C} = -2C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)\mathcal{C} = -2C_{\kappa_1}(r)\mathcal{C} \equiv -2x^0\mathcal{C}\\
&[\mathcal{C}, G_1] = 2S_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)\mathcal{C} = 2S_{\kappa_1}(x)\mathcal{C} = 2S_{\kappa_1}(r)C_{\kappa_2}(\phi)\mathcal{C} \equiv 2x^1\mathcal{C}\\
&[\mathcal{C}, G_2] = 2\kappa_2 S_{\kappa_1\kappa_2}(y)\mathcal{C} = 2\kappa_2 C_{\kappa_1}(x)S_{\kappa_1\kappa_2}(b)\mathcal{C} = 2\kappa_2 S_{\kappa_1}(r)S_{\kappa_2}(\phi)\mathcal{C} \equiv 2\kappa_2 x^2\mathcal{C}.
\end{aligned} \tag{6.4}$$

The operator $\mathcal{C}$ leading to equations (6.3) is written for each specific CK space in table 7 in geodesic parallel I and polar coordinates. Hence, as conformally invariant equations we find

- The usual 2D Laplace equation in $\mathbf{E}^2$ and the corresponding non-zero curvature Laplace–Beltrami versions in the sphere and hyperbolic plane; all of them share the same symmetry algebra $so(3,1)$.
- An equation which does not involve time in the three non-relativistic spacetimes (indeed reducing to a 1D 'Laplace' equation). This agrees with the known absence of a true Galilean invariant wave equation and is the main reason precluding further development of non-relativistic electromagnetic theories [17], where only two separate electric and magnetic essentially static limits are allowed [18, 19].
- The proper (1 + 1)D wave equation is associated with $\mathbf{M}^{1+1}$ [20]; its curvature versions correspond to anti-de Sitter and de Sitter electromagnetism in both $\mathbf{AdS}^{1+1}$ and $\mathbf{dS}^{1+1}$. These three wave-type equations have $so(2,2)$ as their conformal symmetry algebra.

7. Concluding remarks

The present paper gives an approach to the conformal algebras, groups and spaces comprehensive enough to allow a global understanding of those aspects of conformal invariance related to either the space curvature or the metric signature in the initial space. In addition to introducing a new direct derivation for conformal generators, we manage to provide everything in a very explicit form, including the Lie algebra commutators and the differential realizations of the generators of cycle-preserving transformations as first-order differential operators in the CK space $S^2_{[\kappa_1],\kappa_2}$. Thus, we embody, within a single family, all nine spherical, Euclidean, hyperbolic, Galilean, both Newton–Hooke, anti-de Sitter, Minkowskian and de Sitter cases, which are described in a unified form with two parameters linked to the constant curvature and the signature. These results can be taken as a starting point for the study of conformal completion or compactification of spacetimes that will be presented elsewhere.

In both Euclidean and Minkowskian spaces, the transition from the motion to the conformal group can be looked at in two stages: the motion group can be extended first by a one-parameter dilation subgroup, obtaining a similitude group, and then by the specific conformal transformations, closing the whole conformal group. It is a widespread belief that dilation-like transformations do not exist in spaces with non-zero curvature (sphere, de Sitter, ...), but the explicit results obtained here show that this is not so. What is actually different for non-zero curvature is that the intermediate stage provided by similitudes does not exist as a group or Lie algebra, and once a single dilation is added to the motion group, the *full conformal group* is obtained. For $\kappa_1 = 0$ there is a complete symmetry between the translation generators P_i and specific conformal generators G_i, and these are usually introduced as a conjugate to translations by an inversion in the origin. This symmetry *does not* extend to the non-zero curvature case, and while the P_i *do not* commute among themselves when $\kappa_1 \neq 0$, the G_i always commute for any κ_1. There is however a *conformal duality* between translations and Λ-translations, with respective generators P_i and L_i, but this duality is *invisible* in the conformal algebra of flat spaces, where Λ-translations coincide with translations, leaving the specific conformal transformations as a kind of vestigial residue of the difference between P_i and L_i and the symmetry between them as a residue of the deeper duality (5.19).

Finally, we stress that some problems in the conventional Minkowskian quantum field theories may be seen in a new light if dealt with in the non-zero curvature case (anti-de Sitter or de Sitter) taking afterwards a flat limit; in this sense, the study of dependence on the curvature is a natural enquiry. The degeneration of a Lorentzian-type metric produces Newtonian theories, and 'non-relativistic electromagnetic theories', Maxwell–Le Bellac–Lévy–Leblond equations [18, 19], as non-relativistic limits of Maxwell equations also fit inside this parametrized scheme. In this respect, the results here obtained may constitute a starting point for the development of anti-de Sitter and de Sitter electromagnetism.

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Appendix. The lambda function

Let us recall the definitions for the Gudermannian function $\mathrm{gd}(x)$ and the function $\lambda(x)$ [21] which often appear in hyperbolic geometry,

$$\mathrm{gd}(x) = \frac{\pi}{2} - 2\arctan(\mathrm{e}^{-x}) = 2\arctan(\mathrm{e}^{x}) - \frac{\pi}{2} \tag{A.1}$$

$$\lambda(x) = -\mathrm{i}\frac{\pi}{2} - 2\,\mathrm{argtanh}(-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}x}) = 2\mathrm{i}\arctan(\mathrm{e}^{-\mathrm{i}x}) - \mathrm{i}\frac{\pi}{2}. \tag{A.2}$$

They are related by $\lambda(x) = \mathrm{i}\,\mathrm{gd}(-\mathrm{i}x)$ and they are the inverse of each other, $\mathrm{gd}(\lambda(x)) = x$, $\lambda(\mathrm{gd}(x)) = x$. Alternatively, these functions may be defined by the functional relations

$$\tanh\left(\frac{\lambda(x)}{2}\right) = \tan\left(\frac{x}{2}\right) \qquad \tanh\left(\frac{x}{2}\right) = \tan\left(\frac{\mathrm{gd}(x)}{2}\right) \tag{A.3}$$

showing that if $x \in (-\infty, \infty)$, then $\mathrm{gd}(x) \in (-\pi/2, \pi/2)$. If $\mathrm{gd}(x)$ is considered as a point in the circle $\mathbf{S}^1 \equiv (-\pi, \pi]$, the image of $\mathbb{R}$ by the map $\mathrm{gd}(x)$ fills only half the circle. Alternatively, the map $\lambda(x)$ is only defined in half the circle $x \in (-\pi/2, \pi/2)$, and the image is the whole line $\mathbb{R}$.

Within the parametrized CK approach we define the *Lambda function* $\Lambda_\kappa(x)$ as

$$\Lambda_\kappa(x) \equiv x^\wedge := \int_0^x \frac{1}{C_\kappa(t)}\,\mathrm{d}t. \tag{A.4}$$

Hence we find

$$\Lambda_\kappa(x) = \frac{1}{\sqrt{-\kappa}}\frac{\pi}{2} - 2\,\mathrm{arc}\,T_{-\kappa}\left(\frac{1}{\sqrt{-\kappa}}\,\mathrm{e}^{-\sqrt{-\kappa}x}\right) = \begin{cases} \frac{1}{\sqrt{\kappa}}\lambda(\sqrt{\kappa}x) & \kappa > 0 \\ x & \kappa = 0\,. \\ \frac{1}{\sqrt{-\kappa}}\mathrm{gd}(\sqrt{-\kappa}x) & \kappa < 0 \end{cases} \tag{A.5}$$

Therefore, both functions (A.2) and (A.1) are the two particular elliptic or hyperbolic instances of a single CK labelled 'Lambda function'. The analogue of the functional definition (A.3) reads

$$T_{-\kappa}\left(\frac{\Lambda_\kappa(x)}{2}\right) = T_\kappa\left(\frac{x}{2}\right) \qquad T_{-\kappa}\left(\frac{x}{2}\right) = T_\kappa\left(\frac{\Lambda_{-\kappa}(x)}{2}\right) \tag{A.6}$$

where

$$\Lambda_{-\kappa}(\Lambda_\kappa(x)) = x. \tag{A.7}$$

This property extends the known fact that $\lambda(x)$ and $\mathrm{gd}(x)$ are inverse to each other. From this viewpoint, if x is a quantity with label κ, then $\Lambda_\kappa(x)$ is a quantity with label $-\kappa$, and the Lambda function provides a canonical identification between quantities with elliptic and hyperbolic labels. The precise relations required in the main text are

$$\begin{aligned} &C_{-\kappa}(\Lambda_\kappa(x)) = \frac{1}{C_\kappa(x)} \qquad S_{-\kappa}(\Lambda_\kappa(x)) = T_\kappa(x) \qquad T_{-\kappa}(\Lambda_\kappa(x)) = S_\kappa(x) \\ &C^2_{-\kappa}(\Lambda_\kappa(x)) - \kappa S^2_{-\kappa}(\Lambda_\kappa(x)) = 1 \qquad \frac{\mathrm{d}\Lambda_\kappa(x)}{\mathrm{d}x} = \frac{1}{C_\kappa(x)} = C_{-\kappa}(\Lambda_\kappa(x)). \end{aligned} \tag{A.8}$$

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